

Tunneling times through a barrier with inelasticity

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Abstract. Tunneling delay times of wavepackets in quantum mechanical penetration of rectangular barriers have long been known to show a perplexing independence with respect to the width of the barrier. This also has relevance to the transmission of evanescent waves in optics. Some authors have claimed that in the presence of absorption or inelastic channels (which they model by taking a complex barrier potential) this effect no longer exists, in that the time delay becomes proportional to the barrier width. Taking the point of view that complex potentials imply non-Hermitian Hamiltonians and are as such fraught with conceptual pit-falls particularly in connection to problems involving time evolution, we have constructed a two-channel model which does not suffer from such limitations in order to examine this issue. We find that the conclusions arrived at by the earlier authors need to be more precisely specified.

PACS. 03.65.Xp Tunneling, traversal time, quantum Zeno dynamics

QICS. 02.90.+f Foundational issues of quantum mechanics

1 Introduction

The question of tunneling time and the analogous situation of evanescent waves in optics, have in recent years attracted considerable attention. Interest in such matters, however, go back to the forties and fifties of the last century as witnessed by Eisenbud's Ph.D dissertation [1] with E.P. Wigner at Princeton in 1948 and Wigner's own paper [2] in 1955. David Bohm's book on Quantum Theory [3] published in 1951 also sheds light on such issues.

In the present study we shall focus our efforts on one particular aspect of tunneling phenomenon: the counter intuitive conclusion that for a sufficiently opaque barrier tunneling delays are independent of the barrier width, a phenomenon currently referred to as the Hartman [4]-Fletcher [5] effect. By the opaque condition, we imply that the mean energy of the incident wave packet is much smaller than the height of the potential barrier. The tunneling time (τ) is defined in this context as the time taken by the peak of the incident wave-packet to traverse the classically forbidden region and to emerge as the transmitted wave-packet.

Consider the evolution of an incident localized wave-packet described by

$$\int G_{k_0}(k) \exp(ikx - iEt/\hbar) dk$$

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where $G_{k_0}(k)$ is a normalised function of k (say a Gaussian) peaked about the mean momentum $\hbar k_0$. If the wave-packet is allowed to evolve freely the peak would travel with a velocity $\hbar k_0/m$. However with a barrier present, an incident plane wave

$$\frac{1}{\sqrt{2\pi}} e^{ikx - iEt/\hbar}$$

(corresponding to a particle of energy $E = \hbar^2 k^2/2m$, where k is the wave vector) would after transmission through it emerge as

$$\frac{1}{\sqrt{2\pi}} a_T(k) e^{i(kx - Et)/\hbar}$$

where $a_T = |a_T| e^{i\delta}$ is the transmission amplitude (with δ its associated phase). Accordingly, the transmitted wave-packet would be

$$\int G_{k_0}(k) |a_T| e^{i(kx - Et/\hbar + i\delta(k))} dk.$$

Suppose for concreteness, we consider a barrier between $x = 0$ and $x = L$, then the time (τ) at which the peak of the packet will emerge from the barrier at $x = L$ will be given using the method of stationary phase by $d[kL - E\tau/\hbar + \delta(k)]/dk = 0$ or

$$\tau = \hbar \frac{d\delta}{dE} + \frac{L}{(\hbar k/m)}. \quad (1)$$

All quantities are to be evaluated at the peak momentum $\hbar k_0$ and the corresponding energy $\hbar^2 k_0^2/2m$. For notational brevity, we suppress the subscript 0. In our discussion, equation (1) will be taken to be a measure of the tunneling time.

Taking the specific case of a rectangular barrier,

$$V(x) = V_0 \quad \text{for} \quad 0 \leq x \leq L, \quad (2a)$$

$$= 0 \quad \text{elsewhere}, \quad (2b)$$

we solve the Schrödinger equation corresponding to stationary states of energy E ,

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x) = E\psi(x) \quad (3)$$

with the required condition describing reflection and transmission, we have on the two sides of the barrier

$$\psi(x) = e^{ikx} + a_R(k)e^{-ikx} \quad \text{for} \quad x < 0 \quad (4a)$$

$$\psi(x) = a_T(k)e^{ikx} \quad \text{for} \quad x > L, \quad (4b)$$

where $a_R(k)$ and $a_T(k)$ are the reflection and transmission amplitudes whose square moduli give the respective coefficients. The solution within the barrier is given by,

$$\psi(x) = Ae^{qx} + Be^{-qx} \quad (4c)$$

with $q^2 = 2m(V_0 - E)/\hbar^2$. We may as usual proceed to determine A , B , a_R and a_T by matching $\psi(x)$ and $d\psi(x)/dx$ at the boundaries ($x = 0$ and $x = L$) to obtain

$$A = \frac{-(1 + \frac{iq}{k}) \exp(-qL)}{(1 + \frac{q^2}{k^2}) \sinh(qL)} \quad (5a)$$

$$B = \frac{(1 - \frac{iq}{k}) \exp(qL)}{(1 + \frac{q^2}{k^2}) \sinh(qL)} \quad (5b)$$

and

$$a_T = \frac{4i \frac{k}{q} \exp(-ikL)}{\exp(-qL)(1 + i \frac{k}{q})^2 - \exp(qL)(1 - i \frac{k}{q})^2} \quad (6)$$

and accordingly we arrive at the phase-difference δ , viz.,

$$\delta = \tan^{-1} \left[\frac{(q^2 - k^2) \tanh(qL)}{2qk} \right] - kL, \quad (7)$$

and using equation (1), we have

$$\tau = \hbar \frac{d}{dE} \tan^{-1} \left[\frac{k^2 - q^2}{2qk} \tanh qL \right], \quad (8)$$

where the cancellation (in the expression for the time delay) of the contribution arising from the term $-kL$ in δ physically represents the time the wave-packet would have taken to traverse the distance from $x = 0$ to $x = L$ if the potential were absent. It is also understood that we must evaluate the delay at the energy E_0 . We suppress as

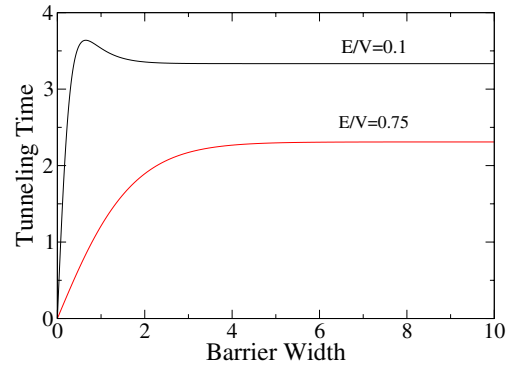


Fig. 1. The tunnel time (τ) is shown as a function of barrier width (L) for two values of E/V_0 namely 0.1 (dashed curve) and 0.75 (solid curve).

mentioned earlier the subscript zero for notational convenience. The tunneling time τ as given by equation (8) is plotted as a function of the barrier width L for two values of V_0/E and shown in Figure 1. As must be the case, $\tau \rightarrow 0$ as $L \rightarrow 0$. But what is more significant is that as $L \rightarrow \infty$ note that $\tanh(qL) \rightarrow 1$ and tunneling time τ become independent of the length L of the barrier. This is the counter-intuitive Hartman [4] Effect. The saturated or asymptotic value of τ is given by

$$\tau_{asympt} = \frac{\hbar}{\sqrt{E(V_0 - E)}} = 2 \frac{m}{\hbar k} \frac{1}{q}, \quad (9)$$

that is, twice the time taken to traverse the decay distance $1/q$ in the barrier region if the particle is moving with its free velocity $\hbar k/m$. Explicitly the expression for τ is given by

$$\tau = \frac{g}{1 + g^2} \left(\frac{m}{\hbar q^2} - \frac{4m}{\hbar(q^2 - k^2)} - \frac{m}{\hbar k^2} - \frac{mL}{\hbar q^2 \sinh(qL) \cosh(qL)} \right)$$

where $g = (q^2 - k^2) \tanh(qL)/2qk$.

It is clearly evident both from the graph and the expression for τ that for large L , τ is independent of L and hence it attains a saturation value. This is what has been known as the Hartman [4]-Fletcher [5] effect for decades. What needs to be emphasized is that the tunneling time approaches the saturation value in two ways depending on the sign of $k^2 - q^2 = 2m(2E - V_0)/\hbar^2$. For $k^2 - q^2 > 0$ (i.e., $E > V_0/2$) it monotonically increases to reach the saturation value and for $k^2 - q^2 < 0$ (i.e., $E < V_0/2$) it reaches the saturation value from above (there is a hump before it attains saturation). Keeping the next to leading term for large L shows the approach to asymptotia viz.,

$$\tau \simeq \tau_{asympt} + 8L \left(\frac{1}{\hbar k/m} \right) \left[\frac{E(V_0 - 2E)}{V_0^2} \right] e^{-2qL}. \quad (10)$$

It is evident from the expression above that there is an exponential approach to the asymptotic limit from below

if $E < V_0/2$ (viz. slope is positive with respect to L) and from above for $E > V_0/2$.

The position of the maximum of the hump (L_0) in the latter case is given by

$$L_0 = \frac{1}{4} \left(\frac{\hbar q}{m} \right) \left(\frac{\hbar}{E} \right) \frac{10E^2 - 9V_0E + V_0^2}{(2E - V_0)(V_0 - E)}. \quad (11)$$

2 Hartman effect in absorbing media

For investigating the Hartman [4]-Fletcher [5] effect in presence of an absorbing media, some authors notably Delgado et al. [6] and Raciti and Salesi [7] phenomenologically introduced an imaginary part to the potential in order to describe absorption. This is in the spirit of the optical model which is often invoked in nuclear physics when there are inelastic channels. However, Delgado et al. do consider the problem from a two-channel point of view as well. The relative phase-shift between the transmitted and the incident wave-packet in such a situation is obtained by simply putting the complex potential form in place of V_0 in q and inserting it in equation (7). If the potential is given by $V = V_0(1 - i\lambda)$ within the range L and zero elsewhere, then the transmission coefficient (a_T) is given by

$$a_T = \frac{4i\frac{k}{q}e^{ikL}}{e^{-qL}(1 + \frac{ik}{q})^2 - e^{qL}(1 - \frac{ik}{q})^2}. \quad (12)$$

Hence in the large L limit, a_T will take the form of

$$a_T = \frac{4i\frac{k}{q}e^{ikL}}{e^{qL}(1 - \frac{ik}{q})^2}. \quad (13)$$

For complex potentials and taking only the first order term in λ , it is seen that

$$q = q_0 \left(1 - \frac{i\lambda V_0}{2(V_0 - E)} \right)$$

where $q_0 = [2m(V_0 - E)/\hbar^2]^{1/2}$. Hence, we have

$$\frac{1}{q} \approx \frac{1}{q_0} \left(1 + \frac{i\lambda V_0}{2(V_0 - E)} \right)$$

which is readily expressed in a polar form as

$$\frac{1}{q} \approx \sqrt{\left(\frac{1}{q_0} \right)^2 + \left(\frac{\lambda V_0}{2q_0(V_0 - E)} \right)^2} e^{i \tan^{-1} \left(\frac{\lambda V_0}{2(V_0 - E)} \right)}. \quad (14)$$

Hence a_T in the presence of the complex potential will be

$$a_T = \frac{-\frac{4ik}{q_0} \left(1 + \frac{i\lambda V_0}{2(V_0 - E)} \right) e^{-ikL - q_0 \left(1 - \frac{i\lambda V_0}{2(V_0 - E)} \right) L}}{\left[1 - \frac{ik}{q} \left(1 + \frac{i\lambda V_0}{2(V_0 - E)} \right) \right]^2}. \quad (15)$$

The only L dependence in the phase-shift comes from the numerator of equation (15) and the dependence is of the form

$$-kL + \frac{q_0 \lambda V_0}{2(V_0 - E)} L.$$

Hence the tunneling time τ for large L is given by

$$\tau \rightarrow \frac{\lambda V_0}{2(V_0 - E)} \frac{k}{q_0} L + \xi \quad (16)$$

where ξ is the L -independent factor that comes from the denominator of equation (15). It is evident from the expression of τ that the tunneling time for large L is not independent of L and hence we no longer have the Hartman-Fletcher effect even for weak absorption (small λ). Optical experiments by Nimtz, Spieker and Brodowsky [8] confirms the absence of the Hartman-Fletcher effect only for strong absorption and the absence of this effect in the presence of weak absorption is not claimed.

Furthermore, one may express doubts regarding the conclusion on tunneling times in the presence of inelastic channels based on the phenomenological device of introducing complex potentials (through the paradigm of the optical model) particularly when we are dealing with questions regarding time evolution. Non-unitary time-evolution using a non-hermitian Hamiltonian could be questionable. If randomness or disorder in the system is invoked then one must use a density matrix approach to the entire problem. We put forward a two-channel formulation for this problem and analyze the matter of tunneling time in this format where unitarity is properly maintained.

To obtain a simple realisation of such a situation, consider the elastic scattering of a particle by a target system describable in terms of a rectangular potential V_0 of width L . Suppose that the target has an excited state of energy Δ above its ground state. When the projectile energy E is above the threshold Δ then, as well as the elastic scattering, an inelastic process is also possible. One may however describe the situation in terms of two channels, the elastic and the inelastic. The latter channel corresponds to the scattering of the projectile of energy $E - \Delta$ from the excited state of the system through a rectangular potential V_i , say, of same width L , while the two channels are coupled to each other via a potential V_c , taken, for simplicity, to be that of same width L . With $\psi(x)$ and $\phi(x)$ denoting the wave-functions in the elastic and the inelastic channels respectively, the system is governed by the coupled Schrödinger equations

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0 \right] \psi(x) + V_c \phi(x) = E\psi(x) \quad (17a)$$

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_i \right] \phi(x) + V_c \psi(x) = (E - \Delta)\phi(x). \quad (17b)$$

The wave-functions for the elastic channel and the inelastic channel in the region $x < L$ are given as

$$\psi(x) = e^{ikx} + a_R(k)e^{-ikx} \quad (18a)$$

$$\phi(x) = R e^{-ik'x} \quad (18b)$$

where $k^2 = 2mE/\hbar^2$ and $k'^2 = 2m(E - \Delta)/\hbar^2$ while $a_R(k)$ and R are the respective reflection amplitudes in the elastic and inelastic channel. Similarly the wave-functions

in the region $x > L$ are

$$\psi(x) = a_T(k)e^{ikx} \quad (19a)$$

$$\phi(x) = Te^{ik'x} \quad (19b)$$

where a_T and T are the respective transmission amplitudes in the elastic and inelastic channels. Note here that naturally there are two outgoing waves to the left and to the right due to the additional inelastic channel as physically required. The wave-functions in the two channels in the intermediate region are

$$\psi(x) = (Be^{\alpha x} + Ce^{-\alpha x}) \sin(\theta/2) + (Fe^{\beta x} + Ge^{-\beta x}) \cos(\theta/2) \quad (20a)$$

$$\phi(x) = (Be^{\alpha x} + Ce^{-\alpha x}) \cos(\theta/2) - (Fe^{\beta x} + Ge^{-\beta x}) \sin(\theta/2) \quad (20b)$$

where

$$\alpha^2 = \left[V_0 - E + \frac{V_0 - V_i - \Delta}{2} (\sec\theta - 1) \right] \quad (21a)$$

$$\beta^2 = \left[V_0 - E + \Delta - \frac{V_0 - V_i - \Delta}{2} (\sec\theta - 1) \right] \quad (21b)$$

$$\theta = \tan^{-1} \left(\frac{2V_c}{V_0 - V_i - \Delta} \right). \quad (21c)$$

The phase difference between the incident and the transmitted waves in the elastic channel can be easily computed from a_T evaluated by demanding the continuity of the wave-function and its derivative at the boundaries. Hence the tunneling time in the elastic channel can be expressed as a function of α and β . The phase difference δ is given as

$$\delta = \tan^{-1} \left(\frac{U - X - Y}{Z} \right) \quad (22)$$

where

$$\begin{aligned} U &= \alpha\beta(k^2 - q^2) [\cosh(\alpha L)\cosh(\beta L) - 1] \\ &\quad \times (\cos 2\theta - 1) \\ X &= [2(\alpha^2 - \beta^2)(k^2 - q^2) \cos\theta - 2(\alpha^2 + \beta^2)(k^2 - q^2)] \\ &\quad \times \sinh(\alpha L)\sinh(\beta L) \\ Y &= [4(\alpha^2 + \beta^2)(k^2 - q^2) \sin^2\theta - 4(k^2 q^2 - \alpha^2 \beta^2)] \\ &\quad \times \sinh(\alpha L)\sinh(\beta L) \\ Z &= 4\alpha k(\beta^2 + q^2)(1 + \cos\theta)\cosh(\alpha L)\sinh(\beta L) \\ &\quad + 4\beta k(\alpha^2 + q^2)(1 - \cos\theta)\cosh(\beta L)\sinh(\alpha L). \end{aligned}$$

The two channels, being coupled by a coupling potential V_c , must emit evanescent wave-functions inside the potential barriers of both channels for the Hartman-Fletcher effect to be present in the elastic channel. As a check, if we put $V_c = 0$, the inelastic channel should decouple and δ should collapse to the old expression of equation (7). Putting $V_c = 0$ and hence $\theta = 0$, we indeed get back the previous expression for δ as in equation (7).

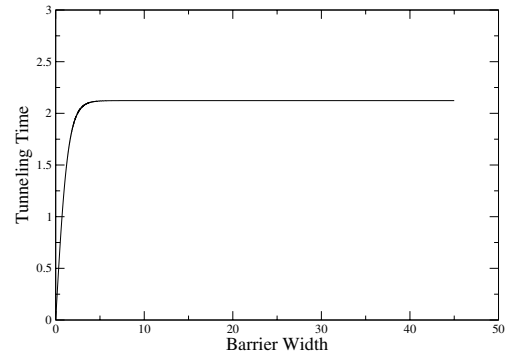


Fig. 2. Tunneling time vs. barrier width for $V_0 = 1.0$, $V_i = 1.0$, $V_c = 0.1$, $E = 0.5$ and $\delta = 0.2$. The presence of the Hartman-Fletcher effect is clearly evident.

To investigate the presence of the Hartman-Fletcher effect in the elastic channel, if we allow L to be large in equation (22) along with the constraint that α and β be real, we see that δ in leading order becomes L independent, viz.,

$$\delta = \tan^{-1} \left(\frac{U_1 - X_1 - Y_1}{Z_1} \right) \quad (23)$$

where

$$\begin{aligned} U_1 &= \alpha\beta(k^2 - q^2)(\cos 2\theta - 1), \\ X_1 &= 2(\alpha^2 - \beta^2)(k^2 + q^2)\cos\theta - 2(\alpha^2 + \beta^2)(k^2 - q^2), \\ Y_1 &= (\alpha^2 + \beta^2)(k^2 - q^2)\sin^2\theta - 4(k^2 q^2 - \alpha^2 \beta^2), \\ Z_1 &= 4\alpha k(\beta^2 + q^2)(1 + \cos\theta) + 4\beta k(\alpha^2 + q^2)(1 - \cos\theta). \end{aligned}$$

Hence, the time delay ($\tau = \hbar d\delta/dk$) will be independent of the width L . We conclude that one observes the Hartman-Fletcher effect whenever the wave-function is evanescent inside both the elastic and inelastic channel, i.e., when both α and β are real. Since

$$\alpha\beta = \sqrt{(V_i - E + \Delta)(V_0 - E) - V_c^2}, \quad (24)$$

one observes the Hartman-Fletcher effect when

$$(V_i - E - \Delta)(V_0 - E) > V_c^2. \quad (25)$$

This is supported by the results of explicit numerical calculation as depicted in Figure 2 which clearly shows the presence of the Hartman-Fletcher effect even in the presence of a coupling potential provided the parameters respect the condition given by equation (25).

It is relevant to confine our attention to the region of parameter space where the elastic channel is 'opaque', viz. $E \ll V_0$. If now the inelastic channel is such that propagation therein is non-evanescent, i.e., $(E - \Delta) > V_i$, then β in the expression for δ given by equation (22) will be purely imaginary. Due to this, the phase difference δ will become L dependent even for large L . It is clear that when we put $\beta = i|\beta|$ in the expression of phase difference in equation (22) and take the limit of large L , we get

$$\delta = \tan^{-1} \left(\frac{U_2 - X_2 - Y_2}{Z_2} \right) \quad (26)$$

where

$$\begin{aligned} U_2 &= \alpha|\beta|(k^2 - q^2)(\cos 2\theta - 1)\cos(|\beta|L) \\ X_2 &= 2(\alpha^2 + |\beta|^2)(k^2 + q^2)\cos\theta \\ &\quad - 2(\alpha^2 - |\beta|^2)(k^2 - q^2)\sin(|\beta|L) \\ Y_2 &= (\alpha^2 - |\beta|^2)(k^2 - q^2)\sin^2\theta \\ &\quad - 4(k^2q^2 + \alpha^2|\beta|^2)\sin(|\beta|L) \end{aligned}$$

and

$$\begin{aligned} Z_2 &= 4\alpha k(-|\beta|^2 + q^2)(1 + \cos\theta)\sin(|\beta|L) \\ &\quad + 4|\beta|k(\alpha^2 + q^2)(1 - \cos\theta)\cos(|\beta|L). \end{aligned} \quad (27)$$

Consequently, the tunneling time for this case will have L dependence even for large L , which reflects the fact that the Hartman-Fletcher effect is absent in this regime.

We have thus demonstrated that the independence of the tunneling time with respect to the width (L) of a barrier for a real potential with large L , claimed to be violated if inelastic channels are present, is a statement which needs to be qualified. We show that if instead of trying to capture inelasticity through the device of a complex potential, we

treat the process through a multichannel formalism (more appropriate for the study of time evolution as compared to approaches using non-Hermitian Hamiltonians), then the outcome depends on specific details of the channels and the strength of the coupling between them.

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